Swami Ramanand Teerth Marathwada University,
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STUDY OF FRACTIONAL DIFFERENTIAL EQUATIONS USING VARIOUS TRANSFORM METHOD

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Submitted by

Mr. Yadav Saiganesh Raju
N.E.S. Science College, Sneh Nagar,
Nanded Dist. Nanded (M.S.), India
Email: ysai6122@gmail.com

Under the supervision of

Dr. R.N. Ingle
M.sc, Ph.D
Professor and Head, Department of Mathematics,
Bahirji Smarak Mahavidyalaya, Basmathnagar,
Dist. Hingoli(M.S), India
Email: ingleraju11@gmail.com

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I. Introduction to proposed research topic:

Fractional differential equations are generalizations of classical differential equations of integer order and have recently been proved to be valuable tools in the modeling of many physical phenomena in various fields of science and engineering. A lot of works have been done by using fractional derivatives for a better description of considered material properties. Mathematical modeling based on enhanced rheological models naturally leads to differential equations of fractional order and to the necessity of the formulation of the initial conditions to such equations. Recently, various analytical and numerical methods have been employed to solve linear and nonlinear fractional differential equations. The differential transform method (DTM) was proposed by Zhou J.K. [1]to solve linear and nonlinear initial value problems in electric circuit analysis. DTM constructs an analytical solution in the form of a polynomial and different from the traditional higher order Taylor series method. For solving two-dimensional linear and nonlinear partial differential equations

Differential equations is a wide field. Many of the laws in Engineering, Economics, Physics, Chemistry & different branches of engineering are based on empirical observations that describe the changes in the states of systems. Mathematical models are often expressed in terms of parameters & their derivatives. initial value problems on linear differential equations with first order and first degree can be solved by many methods.

- Exact Solution
- Newton Rapson Method
- Euler's Method
- Euler's Modified Method
- Runge-Kutta Method
- Taylor Series Method
- Differential Transform Method

Definition of the Riemann Liouville Fractional Integral:

The Riemann-Liouville definition. For $\alpha \in [n-1, n)$, the α-derivative of *f*is

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(\mathrm{n}-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}} dx$$

Definition of the Caputo's Fractional Derivative:

Caputo definition.

For $\alpha \in [n-1, n)$, the α derivative of f is

$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(\mathrm{n}-\alpha)} \int_a^{\alpha} \frac{f^{(n)}(x)}{(\mathrm{t}-\mathrm{x})^{\alpha-\mathrm{n}+1}} d\mathrm{x}.$

Some Important Method .

1) Grunwald-Letnikov method

Another very obvious approach for the discretization of Fractional differential and integral operators is based on straightforward generalization of concepts from classical calculus to the fractional case .specifically, it is well known that an integer order derivative can be written as a differential quotient viz.

$$D^n f(x) = \lim_{h \to 0} \frac{1}{h^n} \sum_{k=0}^{n} (-1)^k \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)} f(x-kh).$$

2) Fractional differential transform method:

There are several approaches to the generalization of the notion of differentiation to fractional orders. The fractional differentiation in Riemann–Liouville sense is defined by

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(\mathrm{n}-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}} dx$$

For $m-1 \leq q < m, m \in Z^+, x > x_0$ let us expand the analytical and continuous function f(x) in terms of a fractional power series as follows

$$f(x) = \sum_{k=0}^{\infty} F(x)(x - x_0)^{k/\alpha}$$

Where α is the order of fraction and F(k) is the fractional differential transform of f(x). concerning the practical applications encounterd in various brances of science, the fractional initial conditions are frequently not available, and it may not be clear what their physical meaning is. Therefore, the definition in above equation should be modified to deal with integer ordered initial conditions in Coputo sense as follows:

$$D_{x_0}^{q}\left[f(x)-\sum_{k=0}^{\infty}\frac{1}{k!}(x-x_0)^{\frac{k}{\alpha}}f^{k}(0)\right]=\frac{1}{\Gamma(m-q)}\frac{d^m}{dt^m}\left\{\int_0^x\left[\sum_{k=0}^{\infty}\frac{1}{k!}(x-x_0)^{\frac{k}{\alpha}}f^{k}(0)\right]dt\right\}$$

Since the initial conditions are implemented to the integer order derivatives, the transformation of the initial conditions are defined as follows:

$$\mathrm{F(k)}=\begin{cases}\mathrm{If}\frac{\mathrm{k}}{\alpha}\in Z^{+}\ then\ \frac{1}{(k/\alpha)}\left[\frac{d^{\frac{k}{\alpha}}f(x)}{dt^{\frac{k}{\alpha}}}\right]_{x=x_0},\ \text{for}k=0,1,2,\dots,(n\alpha-1)\\ \mathrm{If}\frac{\mathrm{k}}{\alpha}\ not\ in\ Z^{+}\ then\ \ 0\end{cases}$$

where, n is the order of FDE considered.

4) Conformable Differential Transform Method

5) Modified Fractional Differential Transform Method

II. Survey or Review of literature and development in the subject area

Firstly, Miller and Ross and then Gray and Zhang introduced discrete versions of the Riemann-Liouville left fractional integrals and derivatives, called the fractional sums and differences with the delta and nabla operators, respectively. In fractional calculus there are two approaches to obtain fractional derivatives. The first approach is by iterating the integral and then defining a fractional order by using Cauchy formula to obtain Riemann fractional integrals and derivatives.

During the review process we had studied a large number of research papers pertaining to the field of study. However, to summarize, some of the researchers and their contributions in the literature are recorded as under:

1. **Zhou, J.K [1] :** In this paper Author applied the differential transform method for solving some classes of Lane-Emden type equations as a model for the of dimensionless density distribution in an isothermal sphere and as a study of the gravitational potential of stars, which are nonlinear ordinary differential equations on the semi-infinte domain.
2. **Ji-Huan. He [2] :** In this paper, author studied a new iteration technique is proposed to solve autonomous ordinary differential systems. In this method, general Lagrange multipliers are introduced to construct correction functionals for the system.
3. **Aytac Arikoglu, Ibrahim Ozakoi [3]**: In this paper differential transform method is extended to solve integro-differential equations. The results obtained by this technique is more accurate as compared to the existing ones.

4. **Aytac Arikoglu, Ibrahim Ozkoi [4]** : In their research paper 'Solution of difference equations by using differential transform method' presents semi numerical analytical solutions to both linear and non-linear difference equations with variable co-efficient by DTM and compared with exact solutions.
5. **Aytac Arikoglu and Ibrahim Ozkol [5] :** In their research paper 'Solution of difference equations by using differential transform method' presents semi numerical analytical solutions to both linear and non-linear difference equations with variable co-efficient by DTM and compared with exact solutions.
6. **Hong Liu and Yongzhong Song [6] :** In this paper, author proved that numerical solution of linear differential-algebric equations (DAEs) is considered by differential transform method. Two index-1 DAEs have been solved by using the method and the numerical solutions coinside with the exact solution very well.
7. **Shaher Momani, Zaid Odibat, Vedat Suat Erturk [7] :**In this paper, author proved that new generalization of the two-dimensional differential transform method that will extend the application of the method to a diffusion-wave equation with space -and time -fractional derivatives.
8. **Narhari Patil and Avinash Khambayat[8] :**In this paper, author studied Differential Transform Method is applied for solving system of Linear Differential Equations. The approximate solution of the equation is calculated in the form of series with easily computable componant.
9. **Salah Abuasad [9]:** In this study, we introduce a new modification of fractional reduced differential transform method (m-FRDTM) to find exact and approximate solutions for nonhomogeneous linear multitier time-fractional diffusion equations (MT-TFDEs) of constant coefficients in a bounded domain with suitable initial conditions.
10. **Mustafa Mamat[10]:**In this paper, author studied the results of the chain rules in conformable fractional calculus and proposed a conformable fractional differential transform method with its applications to fractional derivative.
11. **Hytham. A. Alkresheh [11]:** In this study the multistep fractional differential transform method is employed to obtain approximate analytical solutions of stiff systems of fractional order. The fractional derivative is described in Caputo sense**.**
12. **Pankaj Kumar Srivastava [12] :**In this paper, author discussed numerical solutions of differential equations including the evolution, progress and types of differential equations, special attention is given to the solution of differential equations by application of spline functions.
13. **H. Jafari, S. Sadeghi and Biswas [13] :** In this work, author proved that an analytical solution of linear and nonlinear multidimensional partial differential equation is deduced by the Differential Transform Method.
14. **B. Soltanalizadeh [14] :**In this research, author has been utilized to solve fourth-order parabolic partial differential equations. This method can be used to obtain the exact solutions of this equation.
15. **Sita Charkit [15] :** In this paper, author shows that a simple new method of solving first and second order nonlinear differential equations.
16. **Montri Thongmoon, Sasitorn Pusjuso [16] :** In this paper, author presents the definition and operation of the one-dimensional differential transform and investigate the particular exact solutions of system of ordinary differential equations that usually arise in mathematical biology by a one-dimensional differential transform method.

17. **B. Soltanalizadeh [17]:** In this research, author has been utilized to solve fourth-order parabolic partial differential equations. This method can be used to obtain the exact solutions of this equation.

III. Objectives of Proposed Research Work:

The present research work has the following objectives.

To study the comparisons between different methods of solving D.E. & to search modifications.

(1) To obtain solution of ODE by DTM method & compare solutions with previous research methods.

(2) Presentation of graph of solution of first order & first degree.

(3) Significance of the DTM method for higher order LDE.

(4) To check applicability of DTM method for total differential of complex variable.

IV. Research Methodology:

The researcher has made the use of secondary data which is available in reference books, various journals, research articles, internet, e-library, Swayam (by Govt. of India) based on related to our research topic. The national & international conferences will guide to obtain concrete to proposed research problems. By reading different books, research papers journals, critical reviews of literatures and analyzing them. Try to solve some different problems by different methods.

The methodology consists of the following steps:

1. A literature survey related to topic linear differential equation & ordinary differential equation
2. Researcher will select different (BVP) problems from linear differential equation with initial values & solve by Euler's Method.
3. Researcher will take same problems (IVP) from D.E. & solve by R.K. Method.
4. Researcher will take some (IVP) from DE & solve by Laplace Transform Method.
5. The researcher will solve same IVP on DE by Zhon's method (DTM).
6. The solution of IVP/BVP of D.E. obtained by applying different methods will be compared & present graph of solution whenever possible.
7. To check applicability of DTM for total differential of complex variables or not.

V. Importance of the Study:

Fractional calculus is becoming a useful and key tool in the analysis of scientific problems in Physics, Engineering, Biology and Economics, Fractional formalism to be useful in solving practical problems, fractional calculus, integration and differentiation of an arbitrary or fractional order, provides new tools that expand the descriptive power of calculus beyond the familiar integer-order concept of rate of change and area under thecurve.

a) International Status: Many scientists and researchers many more contributed in the development of fractional differential equations and Fractional calculus like, Leibniz, Euler, Lagrance, Laplace, Fourier, Abel, Liouvielli, Grunwald, Riemann.

Zaid M. Odibat, et.al. : proved that DTM is an analytical and numerical method for solving a wide variety of differential equations X Lotka - Volterra system &chen system, Lorenz system.

B. Soltanalizadeh (2012):proved that the Differential Transformation Method (DTM) has been used to solve fourth order parabolic partial differential equations. This method can be used to obtain exact solutions of this equation. This method can be alternative way for the solution of partial differential equations that have no analytic solutions.

1. **National Status:** Indian's also contributed remarkable in the development of fractional calculus like R.K. Sexana, V.Srivastava, K.N. Rai, S. Das, K. Gupta, S. Saha Roy, V.B.L. Chawrasia, D. Kumar, Pradeep Kumar, D. Bahuguna, Aalam, Suneel Kumar, , B.C. Dhage from Maharashtra, D.N Chaliszar from Gujrat, and Narhari Patil and AvinashKhambayat proved that 'Differential Transform Method for System of Linear Differential Equations' they have studied solution of linear Differential Equation by Laplace Transform Method as exact solution and the numerical result obtained by DTM are compared and found to be close to each other and many more from India had worked on differential transform method in the system of fractional differential equation.

- **Scope of Study:**

The analysis of different methods for solving system of fractional differential equation . During the last decade, fractional calculus has been applied to almost every field of Science, Engineering and Mathematics. Many application of fractional calculus can be found in turbulence and fluid dynamics, stochastic dynamical system, plasma physics and controlled thermonuclear fusion. There are few different methods for obtaining the solution of fractional

differential equation therefore, we develop new different methods to obtain their solutions and discuss the stability and convergence of the numerical methods upto the length. We will study the powerful mathematical software Mathematica to simulate the solutions graphically. We will develop iterative and finite difference schemes to solve them. Solutions of these equations will be simulated graphically. We intend to publish the work in reputed Journals.

- **Limitation of the Work:**

1) The fractional order model is more adequate than integer order model.
2) Fractional derivatives and integrals enable the description of memory properties of various materials and processes.
3) Most of the fractional differential equations do not have analytical solutions.

- **Expected Outcome:**

1) We will try to analyze the development in new results and iterative methods for fractional differential equations.
2) We will try to analyze the numerical methods for fractional differential and Partial differential equation.
3) To find the approximate solutions of fractional differential and difference equation by old and new technique and it will have represented graphical by different softwares like Mathematica, Matlab, Python, Scilab, Mathcad.

- **Year wise plan of work and targets to be achieved:**

Sr.No.	Period	Plan of the work	Targets to be achieved
1.	First year	• To survey the number of reputed research papers in the system fractional differential equation. • To obtain the solutions of initial and boundary value problems of fractional differential and difference equation • To visualize solutions by graphs using a powerful software Mathematica and Matlab.	Excellent understanding of the subject and well-organized course work.
2.	Second year	• To study fractional calculus. • To study fractional differential and difference equations. • To develop and analyze the iterative methods for obtain the solution of fractional differential and difference equations. • To discuss the stability and convergence of iterative methods.	Attempt to obtain fresh results based on existing work.

3.	Third year	• To develop Numerical methods for system of fractional differential equations. • To analysis of development in the different numerical methods for solving fractional differential and difference equations. • To discuss stability and convergence of the numerical methods. • We will publish our work in reputed journals. • Submission of completed work to concerned authority.	Submit atleast three papers.

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Mr. Yadav Saiganesh Raju **Dr. R.N. Ingle**

(Research Scholar) **(Research Guide)**